

GROUP ACTIONS ON GRAPHS AND C^* -CORRESPONDENCES

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ABSTRACT. If G acts on a C^* -correspondence \mathcal{H} (see Definition 2.1), then by the universal property G acts on the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{H}}$ and we study the crossed product $\mathcal{O}_{\mathcal{H}} \rtimes G$ and the fixed point algebra $\mathcal{O}_{\mathcal{H}}^G$. Using intertwiners, we define the Doplicher-Roberts algebra \mathcal{O}_{ρ} of a representation ρ of a compact group G on \mathcal{H} and prove that $\mathcal{O}_{\mathcal{H}}^G$ is isomorphic to \mathcal{O}_{ρ} . When the action of G commutes with the gauge action on $\mathcal{O}_{\mathcal{H}}$, then G acts also on the core algebras $\mathcal{O}_{\mathcal{H}}^{\mathbb{T}}$, where \mathbb{T} denotes the unit circle. We give applications for the action of a group G on the C^* -correspondence \mathcal{H}_E associated to a directed graph E . If G is finite and E is discrete and locally finite, we prove that the crossed product $C^*(E) \rtimes G$ is isomorphic to the C^* -algebra of a graph of C^* -correspondences and stably isomorphic to a locally finite graph algebra. If $C^*(E)$ is simple and purely infinite and the action of G is outer, then $C^*(E)^G$ and $C^*(E) \rtimes G$ are also simple and purely infinite with the same K -theory groups. We illustrate with several examples.

1. INTRODUCTION

Suppose the group G acts on a directed (topological) graph E . This means that G acts on the vertex space E^0 and on the edge space E^1 , preserving incidences. By duality, we get an action of G on the C^* -algebra $C_0(E^0)$ and on the space $C_c(E^1)$, which extends to the $C_0(E^0) - C_0(E^0)$ C^* -correspondence \mathcal{H}_E . In particular, there is a homomorphism $\rho : G \rightarrow \mathcal{L}_{\mathbb{C}}(\mathcal{H}_E)$ into the set of invertible \mathbb{C} -linear operators, called a representation of G on \mathcal{H}_E . By the universal property, this determines an action of G on the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{H}_E}$, also called the graph C^* -algebra and denoted $C^*(E)$. For example, if G finite acts on the graph with one vertex and n loops, then we get an n -dimensional representation $\rho : G \rightarrow \mathcal{L}(\mathbb{C}^n)$ and an action on the Cuntz algebra \mathcal{O}_n . It is known that the fixed point algebra \mathcal{O}_n^G is isomorphic to the Doplicher-Roberts algebra \mathcal{O}_{ρ} (denoted by \mathcal{O}_G in [6]), which in turn is a full corner in a Cuntz-Krieger algebra (see [26]).

In a more general setting, given a group G acting on a $A - A$ C^* -correspondence \mathcal{H} , our goal is to study the fixed point algebra $\mathcal{O}_{\mathcal{H}}^G$ and the crossed product $\mathcal{O}_{\mathcal{H}} \rtimes G$. We construct the Doplicher-Roberts algebra \mathcal{O}_{ρ} associated to $\rho : G \rightarrow \mathcal{L}_{\mathbb{C}}(\mathcal{H})$ from intertwiners (ρ^m, ρ^n) , where $\rho^n = \rho^{\otimes n}$ is the tensor power representation of G on the balanced tensor product $\mathcal{H}^{\otimes n}$. We prove that in certain cases \mathcal{O}_{ρ} is isomorphic to $\mathcal{O}_{\mathcal{H}}^G$ and strongly Morita equivalent to $\mathcal{O}_{\mathcal{H}} \rtimes G$.

If G is finite and it acts on a discrete graph E , we prove that $C^*(E) \rtimes G$ is isomorphic to the C^* -algebra of a graph of (minimal) C^* -correspondences, constructed using the orbits in E^0 and E^1 and the characters of the stabilizer groups. In the proof we use results about the crossed product of a C^* -correspondence by a group G . As a consequence, $C^*(E) \rtimes G$ is

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strongly Morita equivalent to a graph algebra, so its K -theory can be computed in terms of the incidence matrix. In this case, the action of G commutes with the gauge action of \mathbb{T} on $C^*(E)$, so G acts also on the core AF-algebra $C^*(E)^\mathbb{T}$ and $C^*(E)^\mathbb{T} \rtimes G \cong (C^*(E) \rtimes G)^\mathbb{T}$ is an AF-algebra. We recover some examples of group actions on AF-algebras considered by Handelman and Rossmann, see [14].

The paper is organized as follows. In the first section we define group actions on directed graphs and on C^* -correspondences, and we extend these actions to the associated Cuntz-Pimsner algebras. In the next section we define the Doplicher-Roberts algebra associated to a group action on a C^* -correspondence. We continue with general results about crossed products of C^* -correspondences and graphs of C^* -correspondences. The following section contains the main result about finite group actions on discrete graphs. We conclude with several examples of group actions on graphs and K -theory computations for the crossed product and the fixed point algebra.

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2. GROUP ACTIONS ON GRAPHS AND GRAPH ALGEBRAS

Let $E = (E^0, E^1, r, s)$ be a topological graph. Recall that E^0, E^1 are locally compact Hausdorff spaces and $r, s : E^1 \rightarrow E^0$ are continuous with s a local homeomorphism. Denote by $\mathcal{H} = \mathcal{H}_E$ its C^* -correspondence over $A = C_0(E^0)$, obtained by completing $C_c(E^1)$ with the inner product

$$\langle \xi, \eta \rangle(v) = \sum_{s(e)=v} \overline{\xi(e)} \eta(e), \quad \xi, \eta \in C_c(E^1)$$

and multiplications

$$(\xi \cdot f)(e) = \xi(e)f(s(e)), \quad (f \cdot \xi)(e) = f(r(e))\xi(e).$$

A locally compact group G acts on E if there is a continuous morphism $G \rightarrow \text{Aut}(E)$. This means that G acts on the vertex space E^0 and on the edge space E^1 such that the actions are compatible with the range and source maps r, s . This action can be extended to finite paths $e_1 \cdots e_n \in E^n$ by $g \cdot (e_1 \cdots e_n) = (g \cdot e_1) \cdots (g \cdot e_n)$ and similarly to the set of infinite paths E^∞ .

The group action determines a representation $\rho : G \rightarrow \mathcal{L}_\mathbb{C}(\mathcal{H})$ by invertible \mathbb{C} -linear operators on \mathcal{H} and an action of G on $C_0(E^0)$ by $*$ -automorphisms such that

$$(\rho(g)\xi)(e) = \xi(g^{-1} \cdot e) \quad \text{for } \xi \in C_c(E^1) \quad \text{and} \quad (g \cdot a)(v) = a(g^{-1} \cdot v) \quad \text{for } a \in C_0(E^0).$$

A routine verification shows that these actions are compatible with the inner product and the bimodule structure.

Definition 2.1. We say that a group G acts on the C^* -correspondence \mathcal{H} over the C^* -algebra A if G acts on \mathcal{H} via a homomorphism $\rho : G \rightarrow \mathcal{L}_\mathbb{C}(\mathcal{H})$ such that $\rho(g)$ is invertible for all $g \in G$, G acts by $*$ -automorphisms on A and the following relations are satisfied

$$\langle \rho(g)(\xi), \rho(g)(\eta) \rangle = g \cdot \langle \xi, \eta \rangle,$$

$$\rho(g)(\xi a) = (\rho(g)(\xi)) \cdot (g \cdot a), \quad \rho(g)(a \cdot \xi) = (g \cdot a)(\rho(g)(\xi)).$$

The map ρ is called a representation of G on \mathcal{H} .

Remark 2.2. In particular, a group action on the graph E determines as above a group action on the C^* -correspondence \mathcal{H}_E . Notice that a group action on a C^* -correspondence associated to a directed graph is not necessarily determined by an action on the graph.

Remark 2.3. An action of G on the C^* -correspondence \mathcal{H} determines in a natural way an action on the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{H}}$. If the action of G commutes with the gauge action, we get an action of G on the core algebra $\mathcal{O}_{\mathcal{H}}^{\mathbb{T}}$, the fixed point algebra under the gauge action. In particular, an action on a graph E determines an action on the graph algebra $C^*(E)$ by $g \cdot S_e = S_{g \cdot e}$, where S_e is a generator of $C^*(E)$ for $e \in E^1$, and an action on the core algebra $C^*(E)^{\mathbb{T}}$.

Recall that a discrete graph (E^0, E^1) at most countable is row finite if each vertex receives finitely many edges, and is locally finite if in addition each vertex emits finitely many edges.

Theorem 2.4. (*Kumjian and Pask, [24]*) *If G, E are discrete, the action of G on E is free and E is locally finite, then $C^*(E)^G \cong C^*(E/G)$ and*

$$C^*(E) \rtimes G \cong C^*(E/G) \otimes \mathcal{K}(\ell^2(G)),$$

where E/G is the quotient graph.

This result is inspired from a theorem of Green about group actions on spaces, see [12]. A similar result was proved for free and proper actions of locally compact groups on topological graphs in [4], namely that $C^*(E) \rtimes_r G$ is strongly Morita equivalent to $C^*(E/G)$.

In the same paper [24], Kumjian and Pask showed that if G is abelian and $c : E^1 \rightarrow \hat{G}$ is a cocycle, then this induces an action of G on $C^*(E)$ such that $C^*(E) \rtimes G$ is isomorphic to $C^*(E(c))$, where $E(c)$ is the skew product graph $(\hat{G} \times E^0, \hat{G} \times E^1, r, s)$ with

$$r(\chi, e) = (\chi c(e), r(e)), s(\chi, e) = (\chi, s(e))$$

for $\chi \in \hat{G}$. By diagonalization, the action of G on $C^*(E)$ is equivalent to the action α given by $\alpha_g(S_e) = \langle c(e), g \rangle S_e$, where S_e are the generators of $C^*(E)$.

Remark 2.5. If G abelian acts on the \mathcal{O}_n -graph with $E^1 = \{e_1, e_2, \dots, e_n\}$ and $E^0 = \{v\}$, a cocycle $c : E^1 \rightarrow \hat{G}$ determines a representation ρ of G on $\mathcal{H} = \text{span}\{\xi_1, \xi_2, \dots, \xi_n\}$, where $\rho(g)\xi_i = \langle c(e_i), g \rangle \xi_i$. Conversely, an n -dimensional representation of the abelian group G determines a cocycle on the \mathcal{O}_n -graph with values in \hat{G} .

Remark 2.6. Actions of \mathbb{Z}^l on k -graphs were studied by Farthing, Pask and Sims in [11]. In particular, K -theory computations were done for actions of \mathbb{Z} on a row finite 1-graph with no sources such that the orbit of each vertex is finite and either $K_0(C^*(E))$ or $K_1(C^*(E))$ is trivial.

3. DOPLICHER-ROBERTS ALGEBRAS

The Doplicher-Roberts algebras (denoted by \mathcal{O}_G in [6]) were introduced to construct a new duality theory for compact Lie groups $G \subseteq U(n)$ which strengthens the Tannaka-Krein duality. Let \mathcal{T}_G denote the representation category whose objects are tensor powers of the n -dimensional representation ρ of G defined by the inclusion $G \subseteq U(n)$ and whose arrows are the intertwiners. The C^* -algebra \mathcal{O}_G is identified in [6] with the fixed point

algebra \mathcal{O}_n^G , where \mathcal{O}_n is the Cuntz algebra. If σ_G denotes the restriction to \mathcal{O}_G of the canonical endomorphism of the Cuntz algebra, then \mathcal{T}_G can be reconstructed from the pair $(\mathcal{O}_G, \sigma_G)$. Subsequently, Doplicher-Roberts algebras were associated to any object ρ in a strict tensor C^* -category, see [7], [5].

Suppose that the group G acts on the C^* -correspondence \mathcal{H} over A via the homomorphism $\rho : G \rightarrow \mathcal{L}_{\mathbb{C}}(\mathcal{H})$. Inspired from [6], we consider the tensor power $\rho^n : G \rightarrow \mathcal{L}_{\mathbb{C}}(\mathcal{H}^{\otimes n})$, where $\mathcal{H}^{\otimes n}$ is the balanced tensor product over A , and we define the set (ρ^m, ρ^n) of intertwining operators by

$$(\rho^m, \rho^n) = \{T \in \mathcal{L}_A(\mathcal{H}^{\otimes n}, \mathcal{H}^{\otimes m}) \mid T\rho^n = \rho^m T\}.$$

It follows that the linear span of $\bigcup_{m,n \geq 1} (\rho^m, \rho^n)$ has a natural multiplication and involution, after identifying T with $T \otimes I$, where $I : \mathcal{H} \rightarrow \mathcal{H}$ is the identity map.

Definition 3.1. We define the Doplicher-Roberts algebra \mathcal{O}_ρ associated to the representation $\rho : G \rightarrow \mathcal{L}_{\mathbb{C}}(\mathcal{H})$ as the C^* -closure of $\bigcup_{m,n \geq 1} (\rho^m, \rho^n)$ with the above operations.

Remark 3.2. Let the group G act on the C^* -correspondence \mathcal{H} over A . We have $A^G \subseteq (\rho, \rho)$, where A^G denotes the fixed point algebra. Indeed, if $a \in A^G$, then $a \cdot (\rho(g)\xi) = (g \cdot a) \cdot (\rho(g)\xi) = \rho(g)(a \cdot \xi)$.

Example 3.3. If the group G acts on a C^* -algebra A and $\pi : G \rightarrow U(n)$ is a faithful unitary representation, then $\mathcal{H} = \mathbb{C}^n \otimes A$ has a natural structure of C^* -correspondence over A such that G acts on \mathcal{H} by $\rho(g)(x \otimes a) = \pi(g)x \otimes g \cdot a$. It is easy to check that in this case $\mathcal{O}_\rho \cong \mathcal{O}_\pi \otimes A^G$, where \mathcal{O}_π is the (old) Doplicher-Roberts algebra associated to the representation π .

Theorem 3.4. Let \mathcal{H} be a finite projective C^* -correspondence over A and assume the left multiplication $A \rightarrow \mathcal{L}(\mathcal{H})$ is injective. If G is a compact group acting on \mathcal{H} via $\rho : G \rightarrow \mathcal{L}_{\mathbb{C}}(\mathcal{H})$, then $\mathcal{O}_\rho \cong \mathcal{O}_{\mathcal{H}}^G$.

Proof. Since \mathcal{H} is finite projective and the left multiplication is injective, $\mathcal{O}_{\mathcal{H}}$ is isomorphic to the C^* -algebra generated by the span of $\bigcup_{m,n \geq 1} \mathcal{L}(\mathcal{H}^{\otimes m}, \mathcal{H}^{\otimes n})$ after we identify T with $T \otimes I$ (see [18]). Note that G acts on $\mathcal{L}(\mathcal{H}^{\otimes n}, \mathcal{H}^{\otimes m})$ by $(g \cdot T)(\xi) = \rho^m(g)T(\rho^n(g^{-1})\xi)$ and the fixed point algebra is (ρ^m, ρ^n) , so $\mathcal{O}_{\mathcal{H}}^G$ is isomorphic to \mathcal{O}_ρ . \square

Corollary 3.5. Let E be a topological graph such that \mathcal{H}_E is finite projective and the left multiplication is injective. If G is a compact group acting on E , and $\rho : G \rightarrow \mathcal{L}_{\mathbb{C}}(\mathcal{H}_E)$ denotes the representation, then $\mathcal{O}_\rho \cong C^*(E)^G$.

Moreover, if $C^*(E)$ is simple and purely infinite, G is finite and the action on $C^*(E)$ is (pointwise) outer, then \mathcal{O}_ρ and $C^*(E) \rtimes G$ are simple purely infinite and have the same K -theory, therefore are stably isomorphic.

Proof. The first part follows directly from the theorem. The second part is a consequence of a result of A. Kishimoto and A. Kumjian (see Lemma 10 in [22] and Theorem 3.1 in [19]) : If A is simple and purely infinite, G is discrete and $\alpha : G \rightarrow \text{Aut}(A)$ is an action such that α_g is outer for all $g \in G \setminus \{e\}$, then $A \rtimes_{\alpha} G$ is simple and purely infinite. The stable isomorphism follows from the fact that $\mathcal{O}_\rho \cong C^*(E)^G$ is a full corner in $C^*(E) \rtimes G$ and from classification results of simple purely infinite algebras. \square

Remark 3.6. The natural inclusions determine group homomorphisms

$$K_0(C^*(E)^G) \rightarrow K_0(C^*(E)) \rightarrow K_0(C^*(E) \rtimes G)$$

which for finite graphs give information on the class of the identity in $K_0(C^*(E)^G)$ and $K_0(C^*(E) \rtimes G)$.

Example 3.7. Consider a finite group G acting on the graph E_n with one vertex and $n \geq 2$ edges. We denote by ρ the corresponding representation on $\mathcal{H} = \mathcal{H}_E = \mathbb{C}^n$.

Let \hat{G} denote the set of equivalence classes of irreducible unitary representations, and construct as in [26] a graph with the incidence matrix $B = B(\rho)$, where $B(v, w)$ is the multiplicity of w in $v \otimes \rho$ for $v, w \in \hat{G}$. It is shown in [26] that \mathcal{O}_ρ is a full corner in the Cuntz-Krieger algebra \mathcal{O}_B .

For $G = S_n$ the symmetric group acting by permuting the edges of E_n , we get an outer action on the Cuntz algebra \mathcal{O}_n such that $\mathcal{O}_n \rtimes S_n$ is simple and purely infinite, stably isomorphic to $\mathcal{O}_\rho \cong \mathcal{O}_n^{S_n}$. We also get an action of S_n on the core algebra $\mathcal{O}_n^\mathbb{T} \cong M_{n^\infty}$ such that $M_{n^\infty} \rtimes S_n$ is AF.

For $n = 3$, using the character table of S_3 , it was calculated in [26] that

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix},$$

which gives

$$\begin{aligned} K_0(\mathcal{O}_3 \rtimes S_3) &= K_0(\mathcal{O}_\rho) = K_0(\mathcal{O}_B) \cong \mathbb{Z}, \\ K_1(\mathcal{O}_3 \rtimes S_3) &= K_1(\mathcal{O}_\rho) = K_1(\mathcal{O}_B) \cong \mathbb{Z}, \\ K_0(M_{3^\infty} \rtimes S_3) &\cong \varinjlim (\mathbb{Z}^3, B). \end{aligned}$$

The inclusions $\mathcal{O}_3^{S_3} \hookrightarrow \mathcal{O}_3 \hookrightarrow \mathcal{O}_3 \rtimes S_3$ determine the K_0 -theory maps $\mathbb{Z} \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}$. In particular the action of S_3 on \mathcal{O}_3 does not have the Rokhlin property, since the map $\mathbb{Z} \rightarrow \mathbb{Z}_2$ is not injective (see Theorem 3.13 in [16]) and $\mathcal{O}_3^{S_3}$, $\mathcal{O}_3 \rtimes S_3$ are not isomorphic since the classes of the identity in their K_0 -groups do not coincide.

Remark 3.8. If $R(S_3) \cong K_0(S_3)$ is the representation ring of S_3 , then the matrix B above is determined by the map $R(S_3) \rightarrow R(S_3)$ given by multiplication with the character of the representation ρ (see [13, 14]).

Remark 3.9. An action of a group G on a row-finite (discrete) graph E with no sources induces an action of G on the associated graph groupoid $\mathcal{G} = \mathcal{G}_E$ such that $C^*(E) \rtimes G \cong C^*(\mathcal{G} \rtimes G)$, where $\mathcal{G} \rtimes G$ is the semidirect product groupoid with multiplication

$$(\gamma, g)(g^{-1} \cdot \gamma', g') = (\gamma\gamma', gg'),$$

inverse operation

$$(\gamma, g)^{-1} = (g^{-1} \cdot \gamma^{-1}, g^{-1})$$

and range and source maps

$$r(\gamma, g) = (r(\gamma), e), \quad s(\gamma, g) = (g^{-1} \cdot s(\gamma), e).$$

The unit space of $\mathcal{G} \rtimes G$ is identified with \mathcal{G}^0 .

In particular, for the S_n action above we get an action of S_n on the Cuntz groupoid

$$\mathcal{G}_n = \{(x, p - q, y) \in X \times \mathbb{Z} \times X : \sigma^p x = \sigma^q y\},$$

where $\sigma : X \rightarrow X$ is the shift on the unit space $X = \{1, \dots, n\}^{\mathbb{N}}$ such that

$$\mathcal{O}_n \rtimes S_n \cong C^*(\mathcal{G}_n \rtimes S_n).$$

Example 3.10. Given a finite-dimensional unitary representation ρ of a compact group G , Kumjian, Pask, Raeburn and Renault (see [23]) realize \mathcal{O}_ρ as a corner in a graph C^* -algebra and as a groupoid algebra. The graph has vertices \hat{G} , the set of equivalence classes of irreducible representations, and the groupoid is the reduction of the graph groupoid to the set of infinite paths starting at the trivial representation. It turns out that if ρ takes values in $SU(n)$ and is faithful, then the graph is irreducible and locally finite, in particular \mathcal{O}_ρ is simple. Moreover, if G is an infinite compact Lie group and β_ρ denotes the endomorphism of the representation ring $R(G)$ given by tensoring with ρ , then $K_0(\mathcal{O}_\rho) \cong R(G)/\text{im}(1 - \beta_\rho)$ and $K_1(\mathcal{O}_\rho) = 0$. This last result appeared also in A. Wassermann's thesis [33].

4. GROUP ACTIONS ON C^* -CORRESPONDENCES AND CROSSED PRODUCTS

We will need to allow B - A C^* -correspondences where A and B are not necessarily the same C^* -algebras, so we extend our notion of group action:

Definition 4.1. Given C^* -algebras A, B and a B - A C^* -correspondence \mathcal{H} , an action of a locally compact group G on \mathcal{H} is determined by a homomorphism $\rho : G \rightarrow \mathcal{L}_{\mathbb{C}}(\mathcal{H})$ and actions of G on A and B by $*$ -automorphisms such that

$$\langle \rho(g)(\xi), \rho(g)(\eta) \rangle = g \cdot \langle \xi, \eta \rangle,$$

$$\rho(g)(\xi a) = (\rho(g)(\xi))(g \cdot a), \quad \rho(g)(b \cdot \xi) = (g \cdot b) \cdot (\rho(g)(\xi)),$$

where $\xi \in \mathcal{H}, a \in A, b \in B$.

An action of G on \mathcal{H} determines an action of G on $\mathcal{K}(\mathcal{H})$ given by $g \cdot \theta_{\xi, \eta} = \theta_{g \cdot \xi, g \cdot \eta}$, where $\theta_{\xi, \eta}(\zeta) = \xi \langle \eta, \zeta \rangle$.

Recall that if $A = B$, an action of G on \mathcal{H} determines an action on the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{H}}$ (called quasi-free) and, if the action commutes with the gauge action, an action on the core algebra $\mathcal{O}_{\mathcal{H}}^{\text{T}}$.

Definition 4.2. Suppose the group G acts on the B - A C^* -correspondence \mathcal{H} . The crossed product C^* -correspondence $\mathcal{H} \rtimes G$ is defined as $\mathcal{H} \rtimes G = \mathcal{H} \otimes_{\varphi} (A \rtimes G)$, where $\varphi : A \rightarrow \mathcal{L}(A \rtimes G)$ is the embedding of A in the multiplier algebra of $A \rtimes G$, regarded as a Hilbert module over itself.

Remark 4.3. The crossed product $\mathcal{H} \rtimes G$ becomes a $B \rtimes G$ - $A \rtimes G$ C^* -correspondence after the completion of $C_c(G, \mathcal{H})$ using the operations

$$\langle \xi, \eta \rangle(t) = \int_G s^{-1} \cdot \langle \xi(s), \eta(st) \rangle ds,$$

$$(\xi \cdot f)(t) = \int_G \xi(s)(s \cdot (f(s^{-1}t))) ds,$$

$$(h \cdot \xi)(t) = \int_G h(s) \cdot (s \cdot \xi(s^{-1}t)) ds,$$

where $\xi, \eta \in C_c(G, \mathcal{H})$, $f \in C_c(G, A)$, $h \in C_c(G, B)$. Note that the right and left multiplications are given by convolution, and the inner product formula could be also expressed as

$$\langle \xi \otimes f, \eta \otimes f' \rangle = f^* \langle \xi, \eta \rangle f',$$

where this time $\xi, \eta \in \mathcal{H}$, $f, f' \in C_c(G, A)$ and $f^*(t) = t \cdot f(t^{-1})^*$.

Theorem 4.4. (*G. Hao, C.-K. Ng, [15]*) *Let \mathcal{H} be a C^* -correspondence over A and let G amenable locally compact act on \mathcal{H} . Then*

$$\mathcal{O}_{\mathcal{H} \rtimes G} \cong \mathcal{O}_{\mathcal{H}} \rtimes G.$$

Corollary 4.5. *For G amenable acting on a C^* -correspondence \mathcal{H} such that the action commutes with the gauge action we have*

$$\mathcal{O}_{\mathcal{H} \rtimes G}^{\mathbb{T}} \cong \mathcal{O}_{\mathcal{H}}^{\mathbb{T}} \rtimes G.$$

Corollary 4.6. *Let G be a compact group acting on a topological graph E with C^* -correspondence \mathcal{H}_E . Then*

$$C^*(E) \rtimes G \cong \mathcal{O}_{\mathcal{H}_E \rtimes G}.$$

Example 4.7. Let G be a compact group and let \mathcal{E} be a Hermitian vector bundle over a locally compact space X such that G acts on both \mathcal{E} and X in a compatible way (see [1], section 1.6). Such a vector bundle is called a G -vector bundle, generalizing both ordinary vector bundles (when G is trivial) and G -modules (when X reduces to a point). The set of sections $\Gamma(\mathcal{E})$ becomes in the usual way a C^* -correspondence over $C_0(X)$, and the group G acts on $\Gamma(\mathcal{E})$. In particular, G acts on its Cuntz-Pimsner algebra, which is a continuous field of Cuntz algebras, see [32].

Example 4.8. Let G be compact and let $\rho : G \rightarrow U(n)$ be a unitary representation. This determines an action of G on $\mathcal{O}_{\mathcal{H}} \cong \mathcal{O}_n$ where $\mathcal{H} = \mathbb{C}^n$ and a product type action $\bigotimes_{i=1}^{\infty} \text{Ad} \rho$

on $\mathcal{O}_{\mathcal{H}}^{\mathbb{T}} \cong \bigotimes_{i=1}^{\infty} M_n \cong M_{n^{\infty}}$. We obtain the isomorphisms

$$\mathcal{O}_{\mathcal{H} \rtimes G} \cong \mathcal{O}_n \rtimes G, \quad \mathcal{O}_{\mathcal{H} \rtimes G}^{\mathbb{T}} \cong M_{n^{\infty}} \rtimes G.$$

Remark 4.9. Let G be a compact group and let E be a finite graph with C^* -correspondence \mathcal{H}_E . If G acts on \mathcal{H}_E , then we obtain an action of G on $C^*(E)$. If this action commutes with the gauge action, we get an action on the core algebra $C^*(E)^{\mathbb{T}} \cong \varinjlim A_n$, where A_n have dimension m_n . For a locally representable action as in [13, 14], $K_0(C^*(E)^{\mathbb{T}} \rtimes G)$ is the inductive limit of $K_0(A_n \rtimes G) \cong K_0(G)^{m_n}$, where the inclusion maps are determined by matrices with entries in the representation ring $K_0(G) \cong R(G)$.

Remark 4.10. Note that some actions which permute vertices in a graph with more than one vertex may not induce locally representable actions on the core algebra (see Example 6.3).

5. GRAPHS OF C^* -CORRESPONDENCES AND APPLICATIONS TO FINITE GROUPS ACTIONS ON DISCRETE GRAPHS

Given a discrete graph $E = (E^0, E^1, r, s)$, associate to each vertex $v \in E^0$ a C^* -algebra A_v and to each edge $e \in E^1$ a nondegenerate $A_{r(e)}-A_{s(e)}$ C^* -correspondence \mathcal{H}_e . This way we obtain an E -system of C^* -correspondences or a graph of C^* -correspondences. The C^* -algebra associated to this graph of C^* -correspondences is $\mathcal{O}_{\mathcal{H}}$, where $\mathcal{H} = \bigoplus_{e \in E^1} \mathcal{H}_e$ becomes a C^* -correspondence over $A = \bigoplus_{v \in E^0} A_v$ in a natural way. For more information, see [3], where we discuss systems of C^* -correspondences over k -graphs Λ and we construct a Fell bundle over the path groupoid \mathcal{G}_{Λ} such that its reduced cross-sectional algebra is isomorphic to the C^* -algebra of the Λ -system. Unlike in [3], here we allow graphs with sources and C^* -correspondences which are not full.

Example 5.1. Given a discrete graph E , associate to each vertex the C^* -algebra \mathbb{C} and to each edge the C^* -correspondence \mathbb{C} . This is a graph of C^* -correspondences with associated C^* -algebra isomorphic to $C^*(E)$.

Example 5.2. Let E have one vertex v and one loop e , and let $A_v = \mathbb{C}$, $\mathcal{H}_e = \mathbb{C}^n$. Then the C^* -algebra of this graph of C^* -correspondences is \mathcal{O}_n . More general, if $A_v = A$ and $\mathcal{H}_e = \mathcal{H}$, then we get $\mathcal{O}_{\mathcal{H}}$.

Example 5.3. Consider a C^* -correspondence \mathcal{H} over a unital C^* -algebra A such that A decomposes into a direct sum $A_1 \oplus A_2 \oplus \dots \oplus A_n$. If p_j is the identity of A_j , then \mathcal{H} decomposes into a direct sum $\bigoplus_{i,j} p_i \mathcal{H} p_j$ and we can construct a graph of C^* -correspondences with n vertices $\{v_1, \dots, v_n\}$, by assigning the C^* -algebra A_i at v_i and the A_i-A_j C^* -correspondence $p_i \mathcal{H} p_j \neq 0$ at an edge joining v_j with v_i . If some of these correspondences are trivial, there is no edge between the corresponding vertices.

Recall that if a finite group G acts on a finite or countable set X , then $C_0(X) \rtimes G$ decomposes as a direct sum of crossed products $C(Gx) \rtimes G$ over the orbit space X/G . Since the action on each orbit Gx is transitive, this orbit can be identified with the homogeneous space G/G_x , where G_x is the stabilizer group and G acts on G/G_x by left translation. Moreover, $C(G/G_x) \rtimes G \cong M_{|G_x|} \otimes C^*(G_x)$, a (finite) direct sum of matrix algebras.

Corollary 5.4. *If a finite group G acts on a discrete graph E , then*

$$C_0(E^0) \rtimes G \cong \bigoplus_{E^0/G} M(Gv),$$

where $M(Gv)$ is a finite direct sum of matrix algebras. In particular, $C_0(E^0) \rtimes G$ is strongly Morita equivalent (SME) to a direct sum of finite dimensional abelian C^* -algebras.

To describe the crossed product $C^*(E) \rtimes G$, we first consider the case when $C_0(E^0) \rtimes G$ is abelian.

Proposition 5.5. *Suppose $C_0(E^0) \rtimes G = C_0(V)$ with V finite or countable, and denote by $\{p_t\}_{t \in V}$ the minimal projections in $C_0(V)$. The isomorphism classes of separable nondegenerate C^* -correspondences \mathcal{H} over $C_0(V)$ with $*$ -homomorphism $\varphi : C_0(V) \rightarrow \mathcal{L}(\mathcal{H})$ correspond to matrices $(a_{st})_{s,t \in V}$ where a_{st} are nonnegative integer entries or $a_{st} = \infty$. More precisely, $a_{st} = \dim \varphi(p_s) \mathcal{H} p_t$.*

Proof. See Theorem 1.1 in [17]. \square

Corollary 5.6. *If $C_0(E^0) \rtimes G = C_0(V)$ is abelian, then $C^*(E) \rtimes G$ is the graph algebra with incidence matrix $(a_{st})_{s,t \in V}$. It is also the C^* -algebra of a graph of C^* -correspondences where the vertex algebras are \mathbb{C} (one for each vertex $t \in V$) and the C^* -correspondences are Hilbert spaces of dimension a_{st} .*

Proposition 5.7. *Suppose A and B are SME C^* -algebras with A - B imprimitivity bimodule \mathcal{X} . If \mathcal{H} is a C^* -correspondence over A , then $\mathcal{H}' = \mathcal{X}^* \otimes_A \mathcal{H} \otimes_A \mathcal{X}$ is a C^* -correspondence over B such that $\mathcal{O}_{\mathcal{H}}$ and $\mathcal{O}_{\mathcal{H}'}$ are SME.*

Proof. Let $\mathcal{R} = \mathcal{H} \otimes_A \mathcal{X}$ and let $\mathcal{S} = \mathcal{X}^*$. Then $\mathcal{R} \otimes_B \mathcal{S} \cong \mathcal{H}, \mathcal{S} \otimes_A \mathcal{R} \cong \mathcal{H}'$, so by a theorem in [25] (see also [27]), we get that $\mathcal{O}_{\mathcal{H}}$ and $\mathcal{O}_{\mathcal{H}'}$ are SME. \square

Corollary 5.8. *Given a discrete locally finite graph $E = (E^0, E^1, r, s)$ and a finite group G acting on E , the crossed product $C^*(E) \rtimes G$ is SME to a locally finite graph C^* -algebra, where the number of vertices is the cardinality of the spectrum of $C_0(E^0) \rtimes G$. In particular, the K -theory of $C^*(E) \rtimes G$ and of $C^*(E)^{\mathbb{T}} \rtimes G$ can be computed if we determine the incidence matrix of the graph.*

Proof. We apply the Proposition with $A = C_0(E^0) \rtimes G$ and $B = C_0(V)$. \square

Theorem 5.9. *Given a discrete locally finite graph $E = (E^0, E^1, r, s)$ and a finite group G acting on E , the crossed product $C^*(E) \rtimes G$ is isomorphic to the C^* -algebra of a graph of (minimal) C^* -correspondences, where at each vertex v we associate a matrix algebra $M_{n(v)}$ and at each edge joining w and v we associate $M_{n(v), n(w)}$, the set of rectangular matrices with $n(v)$ rows and $n(w)$ columns.*

Proof. Since the group is finite, the orbits in E^0 and E^1 are finite. We decompose the C^* -correspondence $\mathcal{H}_E \rtimes G$ over the C^* -algebra $C_0(E^0) \rtimes G$. This decomposition is obtained in two stages, from the orbits in E^0 and from the characters of the stabilizer groups. For the first stage, we consider the quotient graph E/G and at each vertex $[v] \in E^0/G$ we associate the C^* -algebra $C(Gv) \rtimes G$, where Gv is the orbit of v in E^0 and at each edge $[e] \in E^1/G$ we associate the $C(Gr(e)) \rtimes G$ - $C(Gs(e)) \rtimes G$ C^* -correspondence $C(Ge) \rtimes G$ of the orbit Ge in E^1 . For the second stage, we decompose each $C(Gv) \rtimes G \cong M_{|Gv|} \otimes C^*(G_v)$ into simple components.

Let $C_0(E^0) \rtimes G \cong \bigoplus_{i=1}^m M_{n(i)}$, where $m \in \mathbb{N} \cup \{\infty\}$ and $M_{n(i)}$ denotes the set of $n(i) \times n(i)$

matrix algebras. Consider now the graph with m vertices and at each vertex v_i we assign the C^* -algebra $M_{n(i)}$. If p_i is the unit in $M_{n(i)}$, whenever $p_i(\mathcal{H}_E \rtimes G)p_j \neq 0$, we decompose this as a direct sum of minimal $M_{n(i)}-M_{n(j)}$ C^* -correspondences. A minimal C^* -correspondence is of the form $M_{n(i), n(j)}$, the set of rectangular matrices with $n(i)$ rows and $n(j)$ columns, with the obvious bimodule structure and inner product. Of course, $M_{n,n} = M_n$ and $M_{n,1} = \mathbb{C}^n$. This decomposition determines the number of edges between v_j and v_i . By construction, it follows that $C^*(E) \rtimes G$ is isomorphic to the C^* -algebra of this graph of C^* -correspondences. \square

Remark 5.10. Given a compact group G , denote by $R(G)$ its representation ring. If G acts on a C^* -algebra A , recall that $K_0(A \rtimes G)$ has a structure of $R(G)$ -module. Indeed, given M a finite dimensional G -module with character χ and N a finitely generated projective

$A \rtimes G$ -module, then $M \otimes N$ has a structure of $A \rtimes G$ -module and we can define the product $[N] \cdot \chi$ as $[M \otimes N]$. In particular, given a finite group G acting on a finite graph E , the groups $K_0(C^*(E) \rtimes G)$ and $K_0(C^*(E)^\mathbb{T} \rtimes G)$ have a structure of $R(G)$ -modules.

Remark 5.11. Nekrashevych (see [28]) studied faithful actions of discrete groups G on the set of finite words $X^* = \bigcup_{n=0}^\infty X^n$ over a finite alphabet X , which are self-similar in the sense that for all $g \in G$ and $x \in X$ there exist unique $y \in X$ and $h \in G$ such that $g \cdot (xw) = y(h \cdot w)$. A self-similar action determines an action of G on the rooted tree T_X with root \emptyset and edges from $w \in X^*$ to wx for $x \in X$. He constructed a C^* -correspondence $M = \bigoplus_{x \in X} C^*(G)$ over the C^* -algebra $C^*(G)$, where the left action is the integrated form of a unitary representation of G in $\mathcal{L}(M)$, defined using the self-similarity condition. Since in our notion of representation $\rho : G \rightarrow \mathcal{L}_\mathbb{C}(\mathcal{H})$ the operator $\rho(g)$ is not A -linear, the C^* -correspondence M is not the C^* -correspondence associated with the (infinite) tree T_X and the Cuntz-Pimsner algebra \mathcal{O}_M , denoted also $\mathcal{O}_{(G,X)}$ in [28], is not isomorphic to the crossed product $C^*(T_X) \rtimes G$. Observe though that \mathcal{O}_M contains a copy of the Cuntz algebra \mathcal{O}_n , the tree T_X is the universal covering of the graph E_n with one vertex and $n = |X|$ edges, and $M = \mathbb{C}^n \otimes C^*(G)$.

Exel and Pardo in [9], inspired from the Nekrashevych construction, associate a C^* -algebra $\mathcal{O}_{G,E}$ from a countable discrete group G acting on a finite graph E and a one-cocycle $\varphi : G \times E^1 \rightarrow G$ which determines an action of G on the space of finite paths E^* such that $g \cdot (\alpha\beta) = (g \cdot \alpha)(\varphi(g, \alpha) \cdot \beta)$. This C^* -algebra is defined as the Cuntz-Pimsner algebra of a C^* -correspondence M over $C(E^0) \rtimes G$ and contains a copy of the graph algebra $C^*(E)$. In particular for $G = \mathbb{Z}$ acting on a graph E with $N \times N$ incidence matrix A by fixing the vertices and permuting the edges in a way determined by another $N \times N$ integer matrix B (which also determines the cocycle), they prove that $\mathcal{O}_{G,E}$ is isomorphic to Katsura's algebra $\mathcal{O}_{A,B}$, see [21], used to model all Kirchberg algebras. Note that in this case $C(E^0) \rtimes G$ is isomorphic to the direct sum of N copies of $C(\mathbb{T}) \cong C^*(\mathbb{Z})$.

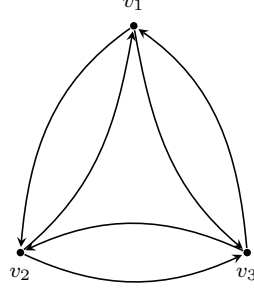
The relationship between the C^* -algebras studied by these authors and the crossed products $C^*(E) \rtimes G$ remains to be explored.

6. EXAMPLES

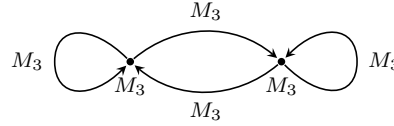
Example 6.1. Let $G = \mathbb{Z}_n$ act freely on its cyclic Cayley graph E with n vertices and n edges. We get a representation ρ of \mathbb{Z}_n on $\mathcal{H} = \mathbb{C}^n$ and an action of \mathbb{Z}_n on $A = \mathbb{C}^n$ which permutes cyclically the basis. In this case $\mathcal{L}_A(\mathcal{H}) = \{T \in M_n : T(\xi a) = T(\xi)a\} \cong \mathbb{C}^n$ (diagonal matrices). Moreover, $(\rho, \rho) \cong \mathbb{C}I$ and $\mathcal{O}_\rho \cong C(\mathbb{T})$, since the quotient graph E/G has one vertex and one loop.

The crossed product $\mathcal{H} \rtimes \mathbb{Z}_n \cong M_n$ becomes a C^* -correspondence over $A \rtimes G \cong M_n$, and $C^*(E) \rtimes G \cong M_n \otimes C(\mathbb{T})$. The graph of C^* -correspondences has one vertex and one loop with M_n attached to each.

Example 6.2. Let E be the graph with three vertices v_1, v_2, v_3 and edges connecting each v_i with v_j for $i \neq j$.

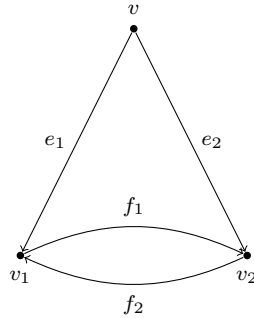


The group $G = S_3$ acts (non-freely) on E by permuting the vertices. There is a single orbit in E^0 and E^1 , and the stabilizer group for each vertex is \mathbb{Z}_2 . The representation ρ is 6-dimensional and $\mathcal{O}_\rho \cong C^*(E)^G \cong \mathcal{O}_2$. We have $C^*(E) \cong M_3(\mathcal{O}_2) \cong \mathcal{O}_2$, so we get an action of S_3 on \mathcal{O}_2 . Here $A = \mathbb{C}^3$ and $A \rtimes S_3 \cong M_3(C^*(\mathbb{Z}_2)) \cong M_3 \oplus M_3$. The C^* -correspondence $\mathcal{H} \rtimes G$ over $M_3 \oplus M_3$ decomposes into $M_3 \oplus M_3 \oplus M_3 \oplus M_3$. The graph of C^* -correspondences is



and $C^*(E) \rtimes G$ is stably isomorphic to \mathcal{O}_2 .

Example 6.3. Consider the graph E with three vertices v, v_1, v_2 and four edges e_1, e_2, f_1, f_2 like in the figure.

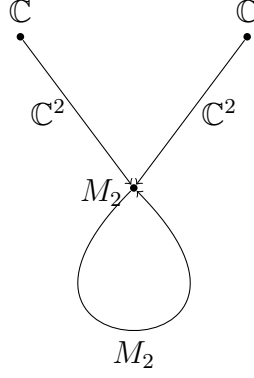


The group $G = \mathbb{Z}_2$ acts on E by fixing v and interchanging v_1 and v_2 . Since the set $\{v\}$ is hereditary and saturated, $C^*(E)$ has an ideal isomorphic to \mathcal{K} such that $C^*(E)/\mathcal{K} \cong M_2(C(\mathbb{T}))$. Since E has sources, the K -theory of $C^*(E)$ is computed using Theorem 3.2 in [31] and

$$K_0(C^*(E)) = \mathbb{Z} \oplus \mathbb{Z}_2, \quad K_1(C^*(E)) = 0.$$

We have $A = \mathbb{C}^3$, $\mathcal{H} = \mathbb{C}^4$, $A \rtimes \mathbb{Z}_2 \cong \mathbb{C} \oplus \mathbb{C} \oplus M_2$ and $\mathcal{H} \rtimes \mathbb{Z}_2 \cong \mathbb{C}^8$ which decomposes as $\mathbb{C}^2 \oplus \mathbb{C}^2 \oplus M_2$. The crossed product $C^*(E) \rtimes \mathbb{Z}_2$ is the C^* -algebra of the following graph

of C^* -correspondences



If E' denotes the subjacent graph, there is an extension

$$0 \rightarrow \mathcal{K} \rightarrow C^*(E') \rightarrow \mathcal{T} \rightarrow 0,$$

where \mathcal{T} is the Toeplitz algebra. We have

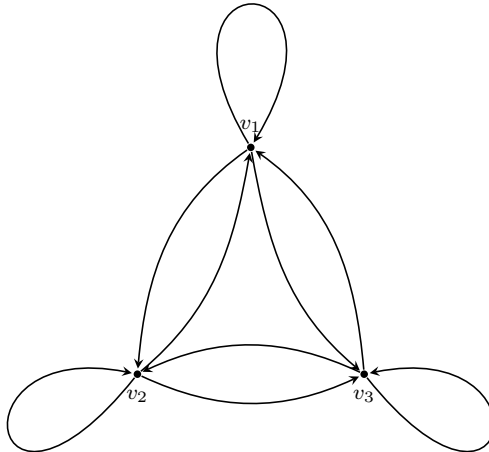
$$K_0(C^*(E) \rtimes \mathbb{Z}_2) = K_0(C^*(E')) = \mathbb{Z}^2, \quad K_1(C^*(E) \rtimes \mathbb{Z}_2) = K_1(C^*(E')) = 0.$$

Example 6.4. Let S_3 act on the graph E with one vertex and three loops by permuting the loops. We get a 3-dimensional representation ρ of S_3 and a non-free action on \mathcal{O}_3 . We already know (see [26]) that $\mathcal{O}_\rho \cong \mathcal{O}_3^{S_3}$ is a full corner in a graph algebra with incidence matrix

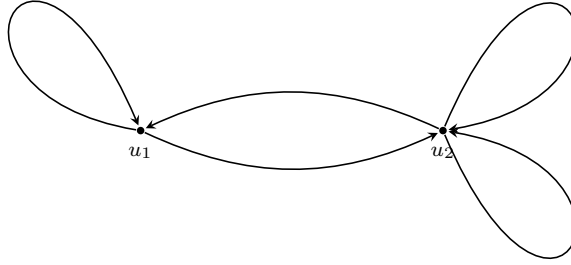
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Since $S_3 \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_2$, we have $\mathcal{O}_3 \rtimes S_3 \cong (\mathcal{O}_3 \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$. We will describe the graph of C^* -correspondences using this iterated crossed product.

It follows that $\mathcal{O}_3 \rtimes \mathbb{Z}_3$ is isomorphic to $C^*(E(c))$, where $c : E^1 \rightarrow \widehat{\mathbb{Z}_3}$ is a cocycle and $E(c)$ is the graph with three vertices v_1, v_2, v_3 and nine edges connecting each v_i with v_j .



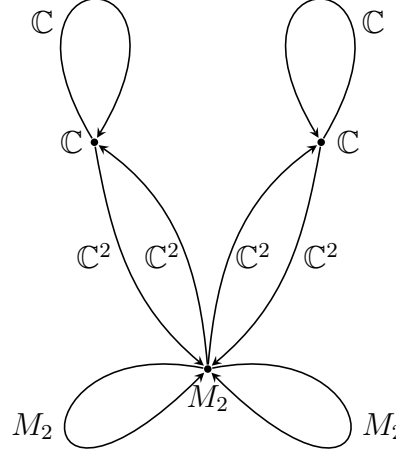
The group \mathbb{Z}_2 acts on $E(c)$ by fixing v_1 and interchanging v_2 with v_3 . If π is the corresponding representation of \mathbb{Z}_2 on \mathbb{C}^9 , it follows that $\mathcal{O}_\pi \cong C^*(E(c))^{\mathbb{Z}_2}$. The quotient graph $E(c)/\mathbb{Z}_2$ has two vertices u_1, u_2 corresponding to the two orbits in $E(c)^0$ and five edges corresponding to the orbits in $E(c)^1$: one loop at u_1 , two loops at u_2 , one edge from u_1 to u_2 and one edge from u_2 to u_1 .



We have $C^*(E(c)) \cong \mathcal{O}_3$, so we get a non-free action of \mathbb{Z}_2 on \mathcal{O}_3 . Here $A = \mathbb{C}^3$ and $A \rtimes \mathbb{Z}_2 \cong \mathbb{C} \oplus \mathbb{C} \oplus M_2 \cong C^*(S_3)$. Moreover, $\mathcal{O}_3 \rtimes S_3 \cong C^*(E(c)) \rtimes \mathbb{Z}_2$ is the C^* -algebra of the following graph of C^* -correspondences. For the vertex u_1 the C^* -algebra is $C^*(\mathbb{Z}_2) \cong \mathbb{C}^2$ and for u_2 the C^* -algebra is $\mathbb{C}^2 \rtimes \mathbb{Z}_2 \cong M_2$. For the loop at u_1 the C^* -correspondence is \mathbb{C}^2 . For each of the two loops at u_2 we have a copy of M_2 . Finally, for each of the remaining two edges in $E(c)/\mathbb{Z}_2$ we have $\mathbb{C}^2 \oplus \mathbb{C}^2$. Since the vertex u_1 splits in two, the C^* -correspondence $\mathbb{C}^9 \rtimes \mathbb{Z}_2$ over $\mathbb{C}^2 \oplus M_2$ decomposes further as $\mathbb{C} \oplus \mathbb{C} \oplus M_2 \oplus M_2 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2$ and we get the following graph of minimal C^* -correspondences: there are three vertices with C^* -algebras \mathbb{C}, \mathbb{C} and M_2 respectively. The incidence matrix of the graph is

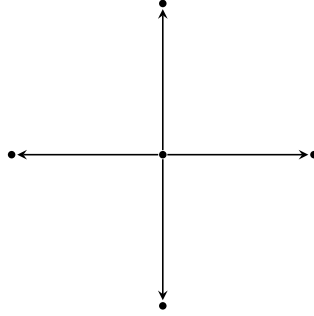
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

and the C^* -correspondences are as in the figure

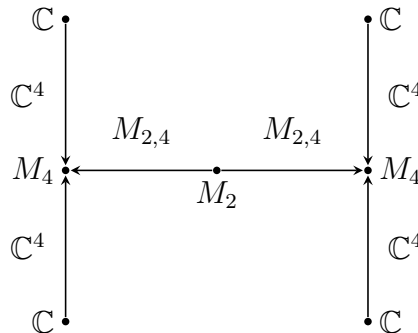


The group S_3 also acts on the core $M_{3\infty}$ of \mathcal{O}_3 . Since $\mathcal{O}_3 \rtimes S_3$ is strongly Morita equivalent to a graph algebra, it follows that its core is strongly Morita equivalent to $M_{3\infty} \rtimes S_3$.

Example 6.5. Let the dihedral group $D_4 \cong \mathbb{Z}_4 \rtimes \mathbb{Z}_2$ act in the natural way on the “cross” graph E with five vertices and four edges by fixing the central vertex.



We get an action of D_4 on $C^*(E) \cong M_5$. It is known that $C^*(E) \rtimes D_4 \cong M_5 \otimes C^*(D_4)$. The vertex space has two orbits and the edge space one orbit. We have $\mathbb{C}^5 \rtimes D_4 \cong C^*(D_4) \oplus \mathbb{C}^4 \rtimes D_4 \cong \mathbb{C}^4 \oplus M_2 \oplus M_4 \oplus M_4$ and $\mathbb{C}^4 \rtimes D_4$ decomposes as $\mathbb{C}^4 \oplus \mathbb{C}^4 \oplus \mathbb{C}^4 \oplus \mathbb{C}^4 \oplus M_{2,4} \oplus M_{2,4}$, obtaining the following graph of C^* -correspondences:



Example 6.6. Let the symmetric group $S_3 = \langle \tau, \sigma \rangle$ act on \mathcal{O}_2 by $\tau(s_1) = s_2, \tau(s_2) = s_1, \sigma(s_1) = \omega s_1, \sigma(s_2) = \omega^2 s_2$, where $\tau = (12), \sigma = (123)$ and $\omega^2 + \omega + 1 = 0$. This action corresponds to the two-dimensional irreducible representation ρ of S_3 , and it commutes with the gauge action. Then $\mathcal{O}_2^{S_3}$ and $\mathcal{O}_2 \rtimes S_3$ have the K-theory of \mathcal{O}_3 , since (see [26]) \mathcal{O}_ρ is a full corner in the algebra of the graph with three vertices and incidence matrix

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

In particular, we get an action of S_3 on the CAR algebra M_{2^∞} and

$$K_0(M_{2^\infty} \rtimes S_3) \cong \varinjlim (\mathbb{Z}^3, B).$$

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